# Assignment 10.

This homework is due *Thursday*, Nov 6.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 5.

## 1. General Lebesgue integral. Quick reminder

For an arbitrary measurable function $f : E \to \mathbb{R} \cup \pm\infty$, define its Lebesgue integral over $E$ by

$$\int_E f = \int_E f^+ - \int_E f^-, \text{ provided at least one of values } \int_E f^+, \int_E f^- \text{ is finite.}$$

In the case when $\int_E f$ is finite (i.e. both $\int_E f^+, \int_E f^-$ are finite) the function $f$ is said to be Lebesgue integrable over $E$.

The integral defined above is linear, monotone and domain additive. Key statements about Lebesgue integral are:

**Fatou's Lemma.** Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $\int_E f \leq \liminf \int_E f_n$.

**Monotone Convergence Theorem.** Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on $E$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $\int_E f = \lim \int_E f_n$.

**The Lebesgue Dominated Convergence Theorem.** Let $\{f_n\}$ be a sequence of measurable functions on $E$. Suppose there is a function $g$ integrable over $E$ s.t. $|f_n| \leq g$ on $E$ for all $n$. If $\{f_n\} \to f$ pointwise a.e. on $E$, then $f$ is integrable over $E$ and $\lim \int_E f_n = \int_E f$.

## 2. Exercises

(1) (4.4.29+) For a locally bounded (therefore bounded on bounded sets by Heine–Borel) measurable function $f$ on $[1, \infty)$, define $a_n = \int_n^{n+1} f$ for each $n \in \mathbb{N}$.
  - (a) Is it true that $f$ is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges?
  - (b) Is it true that $f$ is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely? (*Hint:* Still no.)
  - (c) Is the assertion in the previous item true if we additionally require $f$ to be nonnegative on $[1, \infty)$? (*Hint:* Use the Monotone Convergence Theorem.)

(2)
  - (a) (4.4.34) Let $f$ be a *nonnegative* measurable function on $\mathbb{R}$. Show that
$$\lim_{n\to\infty} \int_{[-n,n]} f = \int_{\mathbb{R}} f.$$
(*Hint:* Use Monotone Convergence theorem.)
  - (b) Prove that the same equality holds if $f$ is arbitrary *integrable* over $\mathbb{R}$ function. (*Hint:* Use the Dominated Convergence.)

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(3) (4.5.37) Let $f$ be integrable function on $E$. Show that for each $\varepsilon > 0$, there is a natural number $N$ for which if $n \geq N$, then $\left|\int_{E_n} f\right| < \varepsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$. (*Hint:* Use continuity of integration; or countable domain additivity of integration.)

(4) (4.5.38i) Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = (-1)^n/n$ for $n \leq x < n+1$, $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} \int_1^n f$ exists while $f$ is not integrable over $[1, \infty)$. Does this contradict continuity of integration?

## 3. Metric spaces. Quick reminder

Metric space is a pair $(X, \rho)$, where $X$ is a nonempty set and $\rho$ is a function $\rho : X \times X \to \mathbb{R}$, called metric, such that $\forall x, y, z \in X$

(1) $\rho(x, y) \geq 0$,
(2) $\rho(x, y) = 0$ if and only if $x = y$,
(3) $\rho(x, y) = \rho(y, x)$,
(4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

A function $\rho$ that satisfies (1),(3),(4) (but not necessarily (2)) is called a *pseudo-metric.*

## 4. More exercises

(5) ($\sim$9.1.4) Let $X = C[a, b]$, the set of all continuous functions on the interval $[a, b]$ of nonzero length. Show that $\rho_1(f, g) = \int_{[a,b]} |f - g|$ is a metric.

(6) ($\sim$9.1.5) Reminder: for sets $A, B$, their *symmetric difference* is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
*The Nikodym Metric.* Let $E$ be a Lebesgue measurable set of real numbers of finite measure. Let $X$ be the set of Lebesgue measurable subsets of $E$, and $m$ Lebesgue measure. For $A, B \in X$ define $\rho(A, B) = m(A \triangle B)$. Show that $\rho$ is pseudometric, but not a metric, on $X$. Show that $\rho(A, B) = \int_E |\chi_A - \chi_B|$.

(7) (a) (9.1.6) Show that for $a, b, c \geq 0$, if $a \leq b + c$, then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$. (*Hint:* Straightforward way: multiply by common denominator; sneaky way: use concavity/convexity of $x/(1 + x)$.)
(b) Let $(X, \rho)$ be an arbitrary metric space. Prove that $(X, \frac{\rho}{1+\rho})$ is also a metric space.
Note. This turns any metric space into a *bounded* metric space.
(c) (9.1.10) Let $\{(X_n, \rho_n)\}$ be a countable collection of metric spaces. Show that $\rho_*$ defines a metric space on the Cartesian product $\prod_{n=1}^{\infty} X_n$, where for points $x = \{x_n\}, y = \{y_n\} \in \prod_{n=1}^{\infty} X_n$,

$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

## 5. Extra Problem

(8) Show that poitwise convergence in $C[0, 1]$ is not metrizable. That is, show that there does not exist a metric $\rho$ on $C[0, 1]$ such that for $f_n, f \in C[0, 1]$, a sequence $\{f_n\}$ converges pointwise to $f$ if and only if $\lim\limits_{n\to\infty} \rho(f_n, f) = 0$.